Axial anomaly and topological charge in lattice gauge theory with overlap Dirac operator

David H. Adams

Math. dept. and Centre for the Subatomic Structure of Matter,
University of Adelaide, S.A. 5005, Australia.

email: dadams@maths.adelaide.edu.au

Abstract

An explicit, detailed evaluation of the classical continuum limit of the axial anomaly/index density of the overlap Dirac operator is carried out in the infinite volume setting, and in a certain finite volume setting where the continuum limit involves an infinite volume limit. Our approach is based on a novel power series expansion of the overlap Dirac operator. The correct continuum expression is reproduced when the parameter m_0 is in the physical region $0 < m_0 < 2$. This is established for a broad range of continuum gauge fields. An analogous result for the fermionic topological charge, given by the index of the overlap Dirac operator, is then established for a class of topologically non-trivial fields in the aforementioned finite volume setting. Problematic issues concerning the index in the infinite volume setting are also discussed.

1 Introduction

In the last few years there have been interesting developments in lattice gauge theory concerning chirality aspects of lattice fermions and topological aspects of lattice gauge fields. The overlap formulation of chiral gauge theories [1, 2], which was largely inspired by Kaplan's domain wall proposal [3], led to the introduction of the overlap Dirac operator for massless lattice fermions [4]. This operator satisfies the the Ginsparg–Wilson relation [5], thereby providing an explicit solution to the chirality problem in lattice QCD [6] (see, e.g., [7] for reviews).

A key test of any lattice formulation of a quantum field theory is whether it reproduces the correct anomalies for the continuum symmetries. In particular, the axial anomaly in QCD explains the $\pi^0 \to 2\gamma$ decay amplitude [8] and should be reproduced in the lattice formulation. Another important quantity in lattice gauge theory is the topological charge of lattice gauge fields. The topological charge enters into an explanation for baryon number violation in the standard model, and for the relatively large value of the η' mass in QCD [9]. Lattice definitions of topological charge is therefore an interesting topic of study. A necessary condition for an acceptable lattice definition of topological charge is that it has the correct continuum limit.

In this paper we study the classical continuum limit of the axial anomaly and fermionic topological charge in lattice gauge theory with fermion action specified by the overlap Dirac operator. Our approach avoids the limitations of earlier approaches which were only valid for small (and hence topologically trivial) background gauge fields. In §3 we develop new techniques (first introduced in [10, (v4)] and [11]) for evaluating the classical continuum limit of the index density/axial anomaly, which enable us to rigorously show that the correct continuum expression is obtained (when $0 < m_0 < 2$) for a broad range of continuum gauge fields. These include topologically non-trivial fields and more general classes of fields which may diverge at infinity or have singularities.

The situation regarding the index is more delicate. In the continuum, in infinite volume, the index of the Dirac operator coupled to an SU(n) gauge field $A(x) = A^a_\mu(x)T^adx^\mu$ on \mathbf{R}^4 is ill-defined in general. A related fact is that the topological charge,

$$Q(A) = \int_{\mathbf{R}^4} d^4x \, q^A(x) \quad , \quad q^A(x) = \frac{-1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr}(F_{\mu\nu}(x) F_{\rho\sigma}(x)) \,, \tag{1.1}$$

is not integer valued,¹ or even well-defined,² in general. However, integer values for Q(A) are guaranteed when A converges sufficiently quickly to a pure gauge configuration at infinity (or at the singularity in the case where A is in a singular gauge). It is in this case that the Dirac operator coupled to A has a well-defined index equal to Q(A) by the Index Theorem (since in this case $A_{\mu}(x)$ corresponds to a gauge potential in an SU(n) bundle over the compact manifold $S^4 \cong \mathbf{R}^4 \cup \{\infty\}$ [12]). Similarly, in the lattice theory the index of the overlap Dirac operator D^U will be ill-defined in general in the infinite volume setting (i.e. infinite lattice on the whole of \mathbf{R}^4). One expects that, in analogy with the continuum case, index D^U will be a well-defined finite number when suitable conditions are imposed on the lattice gauge field U, e.g. a requirement that it converge sufficiently quickly to a pure gauge configuration at infinity. A first step towards clarifying this issue is carried out in §4, where other subtleties/pitfalls concerning index D^U in the infinite volume setting are also discussed.³

To confirm that index D^U really is able to capture topological information about the continuum gauge field it is desirable to verify that index D^U reduces to Q(A) in some setting where the former is a well-defined integer from the beginning and the problems alluded to above do not arise. To do this, in §5 we consider a setting where the continuum field $A_{\mu}(x)$ is in a singular gauge and vanishes outside a bounded region of \mathbf{R}^4 . We restrict the lattice transcript U of A to a large finite volume hypercube;

Examples of fields with arbitrary non-integer topological charge are easily constructed: View \mathbf{R}^4 as $\mathbf{R}^2 \times \mathbf{R}^2$, let (r_1, θ_1) and (r_2, θ_2) be polar coordinates on the two copies of \mathbf{R}^2 ($\theta_j \in [0, 2\pi]$) and choose a smooth function $\lambda(r)$ on \mathbf{R}_+ which vanishes in a neighbourhood of 0 and is equal to a constant c for $r \geq r_0$ (for some finite r_0). Then for any generator T of SU(N) the field $A = \lambda(r_1)Td\theta_1 \pm \lambda(r_2)Td\theta_2$ has topological charge $Q(A) = \mp \mathrm{tr}(T^2)c^2$. This field and its partial derivatives are bounded on \mathbf{R}^4 . Note that the non-integrality of Q(A) does not contradict the results of [12] since the Euclidean action of the field is infinite.

²Even if one allows the values $\pm \infty$, Q(A) is still not well-defined in general since, e.g., the integrand in (1.1) can be oscillatory at infinity.

³In [13] it was erroneously claimed that index $D^U \to Q(A)$ in the classical continuum limit if only $A_{\mu}(x)$ and the derivatives $\partial_{\nu}A_{\mu}(x)$ are bounded. We emphasize that this is not true in general: these conditions are not enough to guarantee that Q(A) is integer or even well-defined, or that index D^U is well-defined, cf. the example above.

then, with (anti)periodic boundary conditions on the lattice spinor fields, index D^U is a well-defined finite integer. We show that, in the continuum limit defined by first taking the infinite volume limit and then the $a \to 0$ limit, index D^U reduces to Q(A). Since index D^U and Q(A) are both integers in this setting it follows that they are also equal close to this continuum limit, i.e. for all sufficiently large volumes and small lattice spacings. This analytic result complements the previous numerical results of [14].

The classical continuum limits of the axial anomaly and topological charge in the finite volume 4-torus case are not considered here; this case is treated in a separate paper [15].

The organisation of this paper is as follows. §2 provides the relevant background material. The material in §3–5 has been described above. (§3: classical continuum limit of the index density/axial anomaly; §4: aspects of index D^U and its continuum limit in the infinite volume lattice setting; §5: classical continuum limit of index D^U starting in a finite volume lattice setting and taking an infinite volume limit.) The conclusions of the paper are summarised in §6.

2 Background

In continuum QCD the axial anomaly arises from the triangle diagram in perturbation theory [8], and can also be understood non-perturbatively as arising from the jacobian for chiral transformations of the fermion fields, regularised by Fujikawa's method [16]. In traditional lattice fermion formulations, such as Wilson-Dirac and staggered fermions, the axial anomaly arises in a different way: The chiral jacobian is trivial, and the anomaly arises instead as a result of the breaking of chiral symmetry by the lattice fermion action; see, e.g., [17, 18]. (The lattice fermion action needs to break chiral symmetry in order to avoid species doubling [19].)

However, a new perspective is possible for the new lattice fermion actions where

the lattice Dirac operator satisfies⁴

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D$$
 (Ginsparg-Wilson relation [5]) (2.1)

$$D^* = \gamma_5 D \gamma_5 \qquad (\gamma_5 - \text{hermiticity})$$
 (2.2)

(a=lattice spacing). Besides the overlap Dirac operator ((2.9) below), another solution is the Dirac operator obtained in the perfect action approach, which is given less explicitly via recursion relations [21]. The fermion action $S = a^4 \sum_x \bar{\psi}(x) D\psi(x)$ is invariant under a new lattice-deformed chiral transformation of the fermion fields [6],⁵

$$\delta\psi = \hat{\gamma}_5\psi \qquad , \qquad \delta\bar{\psi} = \bar{\psi}\gamma_5$$
 (2.3)

where

$$\hat{\gamma}_5 = \gamma_5 (1 - aD) \,, \tag{2.4}$$

It follows from (2.1)–(2.2) that $\hat{\gamma}_5^2 = 1$ and $\hat{\gamma}_5^* = \hat{\gamma}_5$, so $\hat{\gamma}_5$ determines an orthogonal decomposition of lattice spinor fields into "chiral" subspaces just like γ_5 . The fermion measure is not invariant under the new chiral transformation (2.3), and the axial anomaly can be determined from the corresponding jacobian to be $[6]^6$

$$\mathcal{A}(x) = \operatorname{tr}(\hat{\gamma}_5(x, x)) = -ia\operatorname{tr}(\gamma_5 D(x, x)) \tag{2.5}$$

Thus the lattice regularisation with the new lattice fermion actions is providing a lattice version of Fujikawa's non-perturbative perspective on the axial anomaly.

A key question now is whether the axial anomaly (2.5) for the new lattice fermion actions reduces to the usual axial anomaly $\mathcal{A}_{cont}(x) = 2iq^A(x)$ in the classical continuum limit. There have been a number of attempts to show this in the perturbative

⁴ This and the following structure is implicit in the vector overlap formulation; see, e.g., [20].

⁵More generally the action is invariant under $\delta \psi = \gamma_5 (1 - (1 - t)aD)\psi$, $\delta \bar{\psi} = \bar{\psi}\gamma_5 (1 - taD)$ $\forall t \in [0, 1]$. These transformations all lead to the same axial anomaly (2.5).

⁶This is the same as the axial anomaly for the vector overlap, which was considered in a special case in §10 of [2].

framework for general lattice Dirac operator satisfying eq.'s (2.1)–(2.2) [5, 22, 23]. A very general perturbative argument which encompasses these lattice Dirac operators has been given in [24], with the correct continuum anomaly being reproduced provided certain general conditions are satisfied. In the specific case of the overlap Dirac operator an explicit perturbative calculation was carried out in [25]. Subsequently, other calculations of the continuum limit of the axial anomaly for the overlap Dirac operator were carried out in [10, v1],[26],[27]. These did not involve an explicit perturbative expansion in the gauge field; however they did rely on the gauge field being sufficiently small (and hence topologically trivial) so that power series expansions of certain operators involving the gauge field could be carried out. The arguments also required the gauge field to have a well-defined Fourier transformation, which is not always true for general gauge field on \mathbb{R}^4 . Thus the problem of providing a non-perturbative derivation, valid for general classes of gauge fields (including topologically non-trivial fields) still remained. To provide such a derivation is one of the main purposes of the present paper. Our explicit, detailed derivation of the continuum limit of the axial anomaly (in §3) complements an abstract, implicit argument in the case of general gauge field outlined by M. Lüscher in §5 of |28|. (We should also mention that the continuum limit of the axial anomaly in the vector version of the overlap formulation was calculated in [29], and in the domain wall formulation in [30], although these calculations were based on certain approximations (e.g. linearisation of the overlap) and/or assumptions (e.g. weak field, slowly varying field).)

Another appealing feature of lattice Dirac operators satisfying the GW relation (2.1) is that the zero-modes have definite chirality, i.e. the nullspace ker D is invariant under γ_5 : if $D\psi = 0$ then $D(\gamma_5\psi) = (-\gamma_5 D + aD\gamma_5 D)\psi = 0$ [22]. Thus, at least in finite volume settings, index $D = \text{Tr}(\gamma_5|_{\text{ker }D})$ is well-defined. This is in contrast to traditional operators like the Wilson-Dirac operator for which only an approximate (non-integer) index can be defined [17]. Furthermore, there is a "lattice index theorem" [22, 6]:

$$index D = -\frac{a}{2} Tr(\gamma_5 D).$$
 (2.6)

Thus index $D = a^4 \sum_x q(x)$ with the index density

$$q(x) = -\frac{a}{2} \operatorname{tr}(\gamma_5 D(x, x)) \tag{2.7}$$

related to the axial anomaly (2.5) by

$$\mathcal{A}(x) = 2iq(x) \tag{2.8}$$

just as in the continuum. The index of D provides a fermionic definition of the topological charge of the background lattice gauge field. In the case of the overlap Dirac operator⁷

$$D = \frac{1}{a} \left(1 + \gamma_5 \epsilon(H) \right) \qquad , \qquad \epsilon(H) = \frac{H}{\sqrt{H^2}} = -\hat{\gamma}_5$$
 (2.9)

where

$$\frac{1}{a}H = \gamma_5 (D_w - r\frac{m_0}{a}) \tag{2.10}$$

with D_w being the Wilson-Dirac operator and r > 0 the Wilson parameter, the index formula (2.6) gives⁸

$$index D = -\frac{1}{2} Tr \left(\frac{H}{\sqrt{H^2}} \right)$$
 (2.11)

i.e. the spectral asymmetry of $-H = -H_{m_0}$. Since H_m has symmetric spectrum and no zero-modes for m < 0 [1, 2, 14], (2.11) equals minus the spectral flow of H_m coming from the eigenvalue crossings that occur in the interval $0 \le m < m_0$. This is precisely the definition of lattice topological charge arising in the overlap formulation [1, 2]. (The spectral flow definition of lattice topological charge had earlier been studied numerically in [34].) This lattice topological charge is well-defined for all lattice gauge fields except those for which H_{m_0} has zero-modes. After excising this measure-zero subset, the space of lattice gauge fields splits into topological sectors labelled

⁷In fact all solutions of (2.1)–(2.2) have this "overlap" form: D is a solution if and only if $D = \frac{1}{a}(1 + \gamma_5 \epsilon)$ for hermitian $\epsilon = -\hat{\gamma}_5$ with $\epsilon^2 = 1$, cf. Neuberger's first paper in [7].

⁸This is assuming a finite lattice and/or suitable periodicity conditions so that the space of lattice spinor fields is finite-dimensional and hence $\text{Tr}(\gamma_5) = 0$. The role of the relation $\text{Tr}(\gamma_5) = 0$ in this setting has been discussed in detail in [31]; related issues are discussed in [32, 33].

by index D. (This is reminiscent of the way that Lüscher's geometrical definition of lattice topological charge gives a decomposition into topological sectors after excising a measure-zero subset [35].) The physical region for the parameter m_0 , i.e. the values for which D is not afficted with species doubling in the fee fermion case, is $0 < m_0 < 2$ [1, 2, 4]. As we will see, it is precisely for m_0 in this region that the axial anomaly has the correct classical continuum limit.

3 Classical continuum limit of the index density/axial anomaly

In this section we consider the continuum limit of the index density $q^U(x)$ (or equivalently, the axial anomaly $\mathcal{A}(x) = 2iq^U(x)$) for the overlap Dirac operator D^U with U being the lattice transcript of a continuum SU(n) gauge field A on \mathbb{R}^4 .

Put a hypercubic lattice on \mathbf{R}^4 with lattice spacing a. The space of lattice spinor fields (functions on the lattice sites with values in $\mathbf{C}^4 \otimes \mathbf{C}^n$) has the inner product

$$\langle \psi_1, \psi_2 \rangle = a^4 \sum_x \psi_1(x)^* \psi_2(x)$$
 (3.1)

where a contraction of spinor and colour indices is implied. For a given lattice gauge field $U_{\mu}(x)$ the covariant forward (backward) finite difference operators $\frac{1}{a}\nabla_{\mu}^{+}$ $(\frac{1}{a}\nabla_{\mu}^{-})$ are given by

$$\nabla_{\mu}^{+} \psi(x) = U_{\mu}(x) \psi(x + ae_{\mu}) - \psi(x)$$
 (3.2)

$$\nabla_{\mu}^{-}\psi(x) = \psi(x) - U_{\mu}(x - ae_{\mu})^{-1}\psi(x - ae_{\mu})$$
(3.3)

where e_{μ} denotes the unit vector in the positive μ -direction. Set $\nabla_{\mu} = \frac{1}{2}(\nabla_{\mu}^{+} + \nabla_{\mu}^{-})$; this operator is anti-hermitian with respect to the inner product (3.1) since $(\nabla_{\mu}^{\pm})^{*} = \nabla_{\mu}^{\mp}$. The Wilson–Dirac operator is now given by

$$D_w = \frac{1}{a} \nabla + \frac{r}{2} a \left(\frac{1}{a^2} \Delta \right) \tag{3.4}$$

where $\nabla = \sum_{\mu} \gamma^{\mu} \nabla_{\mu}$ (the γ^{μ} 's are taken to be hermitian so ∇ is anti-hermitian), $\Delta = \sum_{\mu} \nabla_{\mu}^{-} + \nabla_{\mu}^{+} = \sum_{\mu} (\nabla_{\mu}^{+})^{*} \nabla_{\mu}^{+} = \sum_{\mu} (\nabla_{\mu}^{-})^{*} \nabla_{\mu}^{-}$ (hermitian, positive) and r > 0 is the Wilson parameter. We then have the hermitian operator

$$H_m = \gamma_5(aD_w - rm) = \gamma_5(\nabla + r(\frac{1}{2}\Delta - m)) \tag{3.5}$$

cf. (2.10). To define the overlap Dirac operator

$$D = \frac{1}{a} \left(1 + \gamma_5 \frac{H}{\sqrt{H^2}} \right) \qquad H = H_{m_0} \tag{3.6}$$

some restrictions must be made on the lattice fields (besides excluding the lattice gauge fields for which H has zero-modes). This can be done in one of the following ways:

- (I) Require $||\psi|| < \infty$. Then the lattice spinor fields form a Hilbert space. Since H is bounded (triangle inequalities give $||H|| \le 8 + 8r + rm_0$), D can be defined via the spectral theory for bounded operators on Hilbert space (see, e.g., [36]). In this setting no restriction on the lattice gauge field $U_{\mu}(x)$ is required (besides the requirement that H has no zero-modes). However, the nullspace ker D need not be finite-dimensional in general, so index D is not defined in general: we can have index $D = \infty \infty$.
- (II) Require $\psi(x)$ and $U_{\mu}(x)$ to be periodic in each direction with fixed periodicity length L. This is equivalent to the finite volume 4-torus setting with topologically trivial gauge fields. The space of such lattice spinor fields is finite-dimensional, and H leaves this space invariant, so D can be defined via the usual spectral theory and has a well-defined finite index.
- (III) More generally, require $\psi(x)$ and $U_{\mu}(x)$ to be related at opposite boundaries of a fundamental domain by a gauge transformation. This is equivalent to the general finite volume 4-torus setting where the gauge fields may be topologically non-trivial. (This was the setting considered in [11].) The space of such lattice spinor fields is finite-dimensional and D is again well-defined with finite index as in (II).
- (IV) Restrict the lattice to a finite hyper-cubic region and require that $\psi(x)$ is periodic (or anti-periodic) at the boundary and $U_{\mu}(x)$ is trivial at the boundary. For such U the operator H leaves the (finite-dimensional) space of these spinor fields invariant, so D is defined in the usual way and has well-defined finite index. This setting is a

variant of (II); the difference is that we do not hold the edge length fixed when taking the (classical) continuum limit. Rather, the limit is taken by first taking the volume to infinity with fixed spacing a, and then taking $a \to 0$ (cf. §5).

In this paper we focus on the settings (I) and (IV). The finite volume torus setting (II)–(III) is treated in a separate paper [15].

In the following the density $\mathcal{O}(x,y)$ of an operator \mathcal{O} on the space of lattice spinor fields is defined through $\mathcal{O}\psi(x) = a^4 \sum_y \mathcal{O}(x,y)\psi(y)$. Equivalently, $\mathcal{O}(x,y) = \frac{1}{a^4} \langle \frac{\delta_x}{a^2}, \mathcal{O} \frac{\delta_y}{a^2} \rangle$ where $\delta_x(y) = \delta_{xy}$. Note that $\{\frac{\delta_x}{a^2}\}$ is an orthonormal basis for the Hilbert space of lattice scalar fields in the infinite volume setting (I). $\mathcal{O}(x,y)$ is a linear operator on $\mathbf{C}^4 \otimes \mathbf{C}^n$, and is well-defined when \mathcal{O} is a bounded operator on the Hilbert space of lattice spinor fields with $||\psi|| < \infty$. The Cauchy–Schwarz inequality gives

$$||\mathcal{O}(x,y)|| \le \frac{1}{a^4} ||\mathcal{O}||. \tag{3.7}$$

In particular, from (2.7) and (2.9) we see that the index density is well-defined:

$$q^{U}(x) = -\frac{a}{2} \operatorname{tr}(\gamma_5 D^{U}(x, x)) = -\frac{1}{2} \operatorname{tr}\left(\frac{H}{\sqrt{H^2}}(x, x)\right)$$
 (3.8)

Our interest is in the classical continuum limit of this quantity, i.e the $a \to 0$ limit when $U_{\mu}(x)$ is the lattice transcript of a continuum gauge field $A_{\mu}(x)$:

$$U_{\mu}(x) = T \exp\left(\int_{0}^{1} aA_{\mu}(x + (1 - \tau)ae_{\mu}) d\tau\right)$$
$$= \sum_{n=0}^{\infty} a^{n} \int_{0 \le \tau_{1} \le \cdots \le \tau_{n} \le 1} d\tau_{n} \cdots d\tau_{1} A_{\mu}(x, \tau_{n}) \cdots A_{\mu}(x, \tau_{1})$$
(3.9)

where $A_{\mu}(x,\tau) = A_{\mu}(x+(1-\tau)ae_{\mu})$ and for simplicity the coupling constant has been set to unity. The technical setup within which the $a \to 0$ limits are taken is as follows. We assume an infinite collection of hyper-cubic lattices on \mathbf{R}^4 has been specified, parameterised by the lattice spacing a, with the following properties: (i) For each $\epsilon > 0$ there are only finitely many lattices with $a \ge \epsilon$. (ii) If a_1 and a_2 are admissible lattice spacings and $a_2 < a_1$ then the lattice parameterised by a_2 is a subdivision of the one parameterised by a_1 , i.e. the sites of the latter are contained in the set of sites

of the former. (E.g. for each a > 0 we have the lattice with sites $a\mathbf{Z}^4$ and can use the collection of these lattices parameterised by $a = a_0, a_0/2, \ldots, a_0/2^p, \ldots$ for some $a_0 > 0$.) The property (ii) implies that if $x \in \mathbf{R}^4$ is a lattice site in some lattice with spacing a then it is also a lattice site in all the other lattices with spacing a' < a. In the following, in statements concerning $a \to 0$ limits the variable x always denotes such a point in \mathbf{R}^4 ; it is fixed in \mathbf{R}^4 and does not change as we go from one lattice to another.

To consider the $a \to 0$ limit we need the overlap Dirac operator $D^U = \frac{1}{a}(1+\gamma_5\frac{H}{\sqrt{H^2}})$ to be well-defined for small a, i.e. H should not have zero-modes when a is sufficiently small. For technical reasons we will furthermore require that H^2 has a strictly positive a-independent lower bound: $H^2 > b > 0$ for sufficiently small a. The existence of such a bound follows from the results of [37] (see also [38]). It was shown there that when $||1-U(p)|| < \epsilon$ for all lattice plaquettes p, where U(p) is the product of the link variables $U_{\mu}(x)$ around p, then there is a lower bound $H^2 > b$, depending only on ϵ and m_0 , such that for fixed $m_0 \in (0,2)$ b > 0 when ϵ is sufficiently small. This result generalises to the case of arbitrary $m_0 \notin \{0,2,4,6,8\}$ [39]. In the present case, where the link variable is given by (3.9), the plaquette variable has the standard expansion

$$1 - U(p_{x,\mu,\nu}) = a^2 F_{\mu\nu}(x) + O(a^3)(x)$$
(3.10)

and hence

$$||1 - U(p)|| \sim O(a^2)$$
 (3.11)

Strictly speaking this bound requires certain conditions on $A_{\mu}(x)$. We will discuss these below, but for the moment we proceed under the assumption that (3.11) is valid, i.e. that there exists a finite K independent of p and small a such that $||1 - U(p)|| < a^2K$. Then, for arbitrary $\epsilon > 0$ we have $||1 - U(p)|| < \epsilon$ for all sufficiently small a, and hence an a-independent lower bound $H^2 > b > 0$ is guaranteed.

To evaluate the continuum limit we use an integral representation for $1/\sqrt{H^2}$ to

expand it in a power series as follows.⁹ We first decompose

$$H^2 = L - V \tag{3.12}$$

where

$$L = -\nabla^2 + r^2(\frac{1}{2}\Delta - m)^2 \tag{3.13}$$

$$V = r \frac{1}{2} \gamma_{\mu} V_{\mu} + \frac{1}{4} [\gamma_{\mu}, \gamma_{\nu}] V_{\mu\nu}$$
 (3.14)

with

$$V_{\mu} = \frac{1}{2} [(\nabla_{\mu}^{+} + \nabla_{\mu}^{-}), \sum_{\nu} (\nabla_{\nu}^{-} - \nabla_{\nu}^{+})]$$
 (3.15)

$$V_{\mu\nu} = \frac{1}{4} [(\nabla_{\mu}^{+} + \nabla_{\mu}^{-}), (\nabla_{\nu}^{+} + \nabla_{\nu}^{-})]$$
 (3.16)

As pointed out in [37], the norms of the commutators of the ∇_{μ}^{\pm} 's are bounded by $\max_{p}||1-U(p)||$. This has the following consequences: First, by (3.11)

$$||V|| \sim O(a^2)$$
. (3.17)

Furthermore,

$$V(x,y) = 0$$
 in regions where the gauge field is pure gauge. (3.18)

The former implies that for small a we have ||V|| < b/2 where b > 0 is the above-mentioned lower bound on H^2 . This in turn implies the lower bound L > b/2 > 0 for the positive operator L in (3.13). Thus for sufficiently small a the operator L is invertible, $||L^{-1}|| \, ||V|| < 1$, and we can make the expansion

$$\frac{H}{\sqrt{H^2}} = H \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \frac{1}{H^2 + \sigma^2}$$

$$= H \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \left(\frac{1}{1 - (L + \sigma^2)^{-1} V} \right) \left(\frac{1}{L + \sigma^2} \right)$$

$$= H \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=0}^{\infty} (G_{\sigma} V)^k G_{\sigma}. \tag{3.19}$$

⁹This expansion was used in Kerler's paper [13] where it was presented as new. In fact it had already been given by the present author in [11].

where $G_{\sigma} := (L + \sigma^2)^{-1}$. Note that the γ -matrices in (3.12) are all contained in V. Since the trace of γ_5 times a product of less than 4 γ -matrices vanishes, the k = 0 and k = 1 terms in (3.19) give vanishing contribution to $q^U(x)$. On the other hand, the terms with $k \geq 3$ satisfy the following bound:

$$\left\| \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=3}^{\infty} [H(G_{\sigma}V)^{k} G_{\sigma}](x,x) \right\|$$

$$\leq \frac{1}{a^{4}} ||H|| \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=3}^{\infty} ||G_{\sigma}||^{k+1} ||V||^{k}$$

$$\leq a^{2} K^{3} ||H|| \left[\int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \frac{1}{(b/2 + \sigma^{2})^{4}} \right] \sum_{k=0}^{\infty} \left(\frac{2}{b} a^{2} K \right)^{k}$$
(3.20)

where we have used (3.7), (3.11) and the bounds $G_{\sigma} < (b/2 + \sigma^2)^{-1} \le 2/b$. This is $O(a^2)$ since the integral and sum are finite and remain so in the $a \to 0$ limit. Hence only the k = 2 term in (3.19) contributes in the $a \to 0$ limit of the index density (3.8), i.e.

$$q^{U}(x) = q_{2}^{U}(x) + O(a^{2})(x)$$
(3.21)

where

$$q_2^U(x) = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \operatorname{tr}[HG_{\sigma}VG_{\sigma}VG_{\sigma}](x,x).$$
 (3.22)

For lattice operators \mathcal{O} which are polynomials in ∇^{\pm}_{μ} we denote by $\mathcal{O}^{(0)}$ the operator obtained by setting U=1 in (3.2)–(3.3). Expanding $U_{\mu}(x)$ in powers of a via (3.9) gives $||H-H^{(0)}|| \sim O(a)$ and $||L-L^{(0)}|| \sim O(a)$. (The rigorous justification of this again requires certain conditions on $A_{\mu}(x)$, to be discussed below.) The latter implies $||G_{\sigma}-G_{\sigma}^{(0)}|| \sim O(a)$. This can be seen in various ways, e.g. as in [10, (v4)], or more simply by noting that $G_{\sigma}-G_{\sigma}^{(0)}=G_{\sigma}^{(0)}(L^{(0)}-L)G_{\sigma}$; this gives $||G_{\sigma}-G_{\sigma}^{(0)}|| \sim O(a)$ due to the above-noted upper bound G_{σ} , $G_{\sigma}^{(0)}<2/b$ which holds for sufficiently small a. This allows us to replace H and G_{σ} by $H^{(0)}$ and $G_{\sigma}^{(0)}$ in (3.22) at the expense of an O(a)(x) term. Furthermore, since $||V|| \sim O(a^2)$ we have $||[L^{(0)},V]|| \sim O(a^3)$, which leads to $||[G_{\sigma}^{(0)},V]|| \sim O(a^3)$ as follows: The bound $||\nabla_{\mu}^{\pm}|| \leq 2$ and triangle inequalities lead to an a-independent upper bound L < c which allows to expand

$$G_{\sigma} = \left(\frac{1}{c+\sigma^2}\right) \left(\frac{1}{1-\frac{c-L}{c+\sigma}}\right) = \frac{1}{c+\sigma^2} \sum_{m=0}^{\infty} \left(\frac{c-L}{c+\sigma^2}\right)^m \tag{3.23}$$

Now, since

$$||[(c-L^{(0)})^m, V]|| \le m||[L^{(0)}, V]|| \cdot ||c-L||^{m-1} \le m(a^3K)(c-b/2)^{m-1}$$
 (3.24)

we get

$$||[G_{\sigma}^{(0)}, V]|| \le \frac{a^3 K}{c^2} \sum_{m=0}^{\infty} (m+1) \left(\frac{c-b/2}{c}\right)^m$$
 (3.25)

and this is $\sim O(a^3)$ since the sum converges (since 0 < b/2 < c). Taking this into account in (3.22), it follows from (3.21) that

$$q(x) = -\frac{1}{2} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \operatorname{tr}[H^{(0)}V^{2}(G_{\sigma}^{(0)})^{3}](x,x) + O(a)(x)$$

$$= -\frac{1}{2} \operatorname{tr}\left[H^{(0)}V^{2} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \frac{1}{(L^{(0)} + \sigma^{2})^{3}}\right](x,x) + O(a)(x)$$

$$= \frac{-3}{16} \operatorname{tr}[H^{(0)}V^{2}(L^{(0)})^{-5/2}](x,x) + O(a)(x)$$
(3.26)

Evaluating the trace over spinor indices we find

$$q^{U}(x) = \frac{-3r}{16} \epsilon_{\mu\nu\rho\sigma} \text{tr} \Big[\left(-\nabla_{\mu}^{(0)} (V_{\nu} V_{\rho\sigma} + V_{\nu\rho} V_{\sigma}) + \left(\frac{1}{2} \Delta^{(0)} - m_{0} \right) V_{\mu\nu} V_{\rho\sigma} \right) (L^{(0)})^{-5/2} \Big] (x, x) + O(a)(x)$$

$$(3.27)$$

where V_{μ} and $V_{\mu\nu}$ are given by (3.15)–(3.16). Noting that [37]

$$[\nabla_{\mu}^{+}, \nabla_{\nu}^{+}]\psi(x) = (1 - U(p_{x,\mu\nu}))U_{\mu}(x)U_{\nu}(x + ae_{\mu})\psi(x + ae_{\mu} + ae_{\nu})$$
(3.28)

and similar formulae for the other commutators, calculations with (3.9) give

$$[\nabla_{\mu}^{\pm}, \nabla_{\nu}^{\pm}]\psi(x) = (a^{2}F_{\mu\nu}(x) + O(a^{3})(x))\psi(x \pm ae_{\mu} \pm ae_{\nu})$$
(3.29)

$$[\nabla_{\mu}^{\pm}, \nabla_{\nu}^{\mp}]\psi(x) = (a^{2}F_{\mu\nu}(x) + O(a^{3})(x))\psi(x \pm ae_{\mu} \mp ae_{\nu})$$
(3.30)

These determine the relevant contributions of V_{μ} and $V_{\mu\nu}$ in (3.27).

We now exploit the fact that the delta-function δ_x on the lattice sites has the Fourier expansion in plane wave fields:

$$\delta_x = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{-ikx/a} \phi_k \tag{3.31}$$

where $\phi_k(y) := e^{iky/a}$. For a general operator \mathcal{O} this leads to

$$\mathcal{O}(x,x) = \frac{1}{a^4} \langle \frac{\delta_x}{a^2}, \mathcal{O}\frac{\delta_x}{a^2} \rangle = \frac{1}{a^4} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{-ikx/a} \frac{1}{a^4} \langle \delta_x, \mathcal{O}\phi_k \rangle$$
$$= \frac{1}{a^4} \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} e^{-ikx/a} (\mathcal{O}\phi_k)(x)$$
(3.32)

In the case where

$$\mathcal{O} = \epsilon_{\mu\nu\rho\sigma} \left(-\nabla_{\mu}^{(0)} (V_{\nu} V_{\rho\sigma} + V_{\nu\rho} V_{\sigma}) + \left(\frac{1}{2} \Delta^{(0)} - m_0 \right) V_{\mu\nu} V_{\rho\sigma} \right) (L^{(0)})^{-5/2}$$
(3.33)

a calculation using (3.15)–(3.16) with (3.29)–(3.30) gives¹⁰

$$(\mathcal{O}\phi_k)(x) = 32\pi^2 a^4 \lambda(k; r, m_0)(q^A(x) + O(a)(x))\phi_k(x)$$
(3.34)

where $q^{A}(x)$ is the continuum index density (1.1), and

$$\lambda(k; r, m) = \frac{\prod_{\nu} \cos k_{\nu} \left(-m + \sum_{\mu} (1 - \cos k_{\mu}) - \sum_{\mu} \frac{\sin^{2} k_{\mu}}{\cos k_{\mu}} \right)}{\left[\sum_{\mu} \sin^{2} k_{\mu} + r^{2} (-m + \sum_{\mu} (1 - \cos k_{\mu}))^{2} \right]^{5/2}}$$
(3.35)

It follows from (3.27) and (3.32) that

$$q(x) = I(r, m_0)q^A(x) + O(a)(x)$$
(3.36)

where

$$I(r,m) = \frac{-3r}{8\pi^2} \int_{-\pi}^{\pi} d^4k \,\lambda(k;r,m) \,. \tag{3.37}$$

The integral I(r, m) is similar to the integral (A.17) of [18], although the exponents and numerical factor are different and the parameter m did not appear there. To evaluate it we exploit the symmetries of the integrand (as in [18]) and change variables to $s_{\nu} \equiv \sin k_{\nu}$ to write

$$I(r,m) = \sum_{\epsilon_{\mu}=\pm 1} \left(\prod_{\mu=1}^{4} \operatorname{sign}(\epsilon_{\mu}) \right) I(r,m,\epsilon)$$
 (3.38)

where

$$I(r, m, \epsilon) = \frac{-3r}{8\pi^2} \int_{-1}^{1} d^4s \frac{-m + \sum_{\mu} (1 - \epsilon_{\mu} \sqrt{1 - s_{\mu}^2}) - \sum_{\mu} \frac{s_{\mu}^2}{\epsilon_{\mu} \sqrt{1 - s_{\mu}^2}}}{\left[s^2 + r^2 \left(-m + \sum_{\mu} (1 - \epsilon_{\mu} \sqrt{1 - s_{\mu}^2}) \right)^2 \right]^{5/2}}$$
(3.39)

¹⁰Essentially the same calculations were presented as new in [13]. In fact they had already been done in [10, (v4)] and [11].

This diverges for $m = m_{\epsilon} \equiv \sum_{\mu} (1 - \epsilon_{\mu}) \in \{0, 2, 4, 6, 8\}$ but is finite for all other values of m. It is a constant function of m with a jump at m_{ϵ} ; to see this set $\widetilde{\Delta}(s, m) = -m + \sum_{\mu} (1 - \epsilon_{\mu} \sqrt{1 - s_{\mu}^2})$, then

$$\frac{-8\pi^2}{3} \frac{d}{dm} I(r, m, \epsilon) = -r \int_{-1}^{1} \frac{d^4s}{(s^2 + r^2 \tilde{\Delta}^2)^{5/2}} + 5r^3 \int_{-1}^{1} d^4s \frac{\tilde{\Delta}(1 - \sum_{\nu} s_{\nu} \frac{\partial}{\partial s_{\nu}}) \tilde{\Delta}}{(s^2 + r^2 \tilde{\Delta}^2)^{7/2}}$$
(3.40)

Inspired by the identity eq. (A.19) of [18] (which was originally due to Karsten and Smit) we rewrite the second integral as

$$5r \int_{-1}^{1} d^{4}s \frac{\left(1 - \frac{1}{2} \sum_{\nu} s_{\nu} \frac{\partial}{\partial s_{\nu}}\right) \left(s^{2} + r^{2} \widetilde{\Delta}^{2}\right)}{\left(s^{2} + r^{2} \widetilde{\Delta}^{2}\right)^{7/2}}$$

$$= 5r \int_{-1}^{1} d^{4}s \left(1 + \frac{1}{5} \sum_{\nu} s_{\nu} \frac{\partial}{\partial s_{\nu}}\right) \left(s^{2} + r^{2} \widetilde{\Delta}^{2}\right)^{-5/2}$$
(3.41)

Integration by parts now gives the first integral in (3.40) except with the opposite sign, so (3.40) vanishes. I(r,m) can now be determined by evaluating $I(r,m,\epsilon)$ in the limits $m \to m_{\epsilon}$ from above and below. For $m > m_{\epsilon}$, after setting $m = m_{\epsilon} + \widetilde{m}$ and changing variables to $\widetilde{s}_{\nu} = s_{\nu}/\widetilde{m}$ in (3.39), we get

$$I(r, m, \epsilon) = \frac{-3r}{8\pi^2} \int_{-1/\widetilde{m}}^{1/\widetilde{m}} d^4 \widetilde{s} \, \widetilde{m}^4 \frac{-\widetilde{m} + O(\widetilde{m}^2)}{\widetilde{m}^5 [\widetilde{s}^2 + r^2(-1 + O(\widetilde{m}))^2]^{5/2}}$$

$$\stackrel{\widetilde{m} \to 0_+}{=} \frac{-3}{8\pi^2} \int_{-\infty}^{\infty} d^4 \widetilde{s} \, \frac{-r}{(\widetilde{s}^2 + r^2)^{5/2}} = 1/2$$
(3.42)

For $m < m_{\epsilon}$ a similar calculation gives $I(r, m, \epsilon) = -1/2$. Thus I(r, m) is independent of r > 0 and can now be calculated from (3.38), leading to the value I(m) in table below.¹¹

$$I(m) = \begin{vmatrix} 0 < m < 2 & 2 < m < 4 & 4 < m < 6 & 6 < m < 8 & m \notin [0, 8] \\ 1 & -3 & 3 & -1 & 0 \end{vmatrix}$$

$$(3.43)$$

By (3.36) we now have

$$q^{U}(x) = I(m_0)q^{A}(x) + O(a)(x)$$
(3.44)

¹¹The evaluation of this integral was given in the first version [10, (v1)] of this paper, and also (in more detail) in [27].

Thus for $0 < m_0 < 2$ the index density $q^U(x)$ does indeed have the correct classical continuum limit in the infinite volume setting.

The $O(a^p)$ bounds used in the calculations above can be established by standard calculations when the continuum field $A_{\mu}(x)$ is smooth and $||A_{\mu}(x)||$, $||\partial_{\mu}A_{\nu}(x)||$ and $||\partial_{\mu}\partial_{\nu}A_{\rho}(x)||$ are all bounded on \mathbf{R}^4 (the details can be found in the appendix of [15]). However, the result $q(x) = q^A(x) + O(a)(x)$ (and its generalisation (3.44) when $m_0 \notin [0,2]$) holds in more general cases as well: It suffices that the plaquette variable of the lattice transcript of A satisfies $||1 - U(p)|| < \epsilon$ when the lattice spacing a is sufficiently small, with ϵ small enough that a strictly positive lower bound $H^2 > b > 0$ is guaranteed (as discussed earlier). Then D^U is local in the gauge field [37], leading to

$$|q^{U}(x) - q^{\widetilde{U}}(x)| \sim O(\frac{1}{a^4} e^{-\rho/a})$$
 (3.45)

where \widetilde{U} is the lattice transcript of a smooth continuum field \widetilde{A} which coincides with A in a neighbourhood of x and vanishes outside a bounded region of \mathbf{R}^4 . Since such \widetilde{A} and its partial derivatives are automatically bounded on \mathbf{R}^4 , we have $q^{\widetilde{U}}(x) = q^A(x) + O(a)(x)$, and this together with (3.45) gives $q(x) = q^A(x) + O(a)(x)$.

A more detailed justification of the preceding is as follows. Let A be an arbitrary smooth continuum gauge field on \mathbf{R}^4 . Pick a smooth function $\lambda(y)$ on \mathbf{R}^4 which is equal to 1 in the box $\prod_{\mu}[x_{\mu}-de_{\mu},x_{\mu}+de_{\mu}]\subset\mathbf{R}^4$ (d>0) and which vanishes outside a bounded region of \mathbf{R}^4 . To prove the claims above, it suffices to establish (3.45) for the case where $\tilde{A}_{\mu}(y)=\lambda(y)A_{\mu}(y)$. For this we exploit the fact [37] that there is a power series expansion $1/\sqrt{H^2}=\kappa\sum_{k=0}^{\infty}t^kP_k(H^2)$ where $P_k(\cdot)$ is a Legendre polynomial of order k; $||P_k(H^2)|| \leq 1$; $t=e^{-\theta}$; the constants $\kappa,\theta>0$ depend only on the (strictly positive) lower and upper bounds on H^2 [37]. Similarly $1/\sqrt{\tilde{H}^2}=\kappa\sum_{k=0}^{\infty}t^kP_k(\tilde{H}^2)$ where \tilde{H} is obtained from H by replacing the lattice gauge field U (the lattice transcript of A) by the lattice transcript \tilde{U} of \tilde{A} . (This requires H^2 and \tilde{H}^2 to have a lower bound b>0. This is guaranteed in both cases when a is sufficiently small: In the first case we have $||1-U(p)|| < \epsilon$ by assumption, while in the second case $||1-\tilde{U}(p)|| \sim O(a^2)$ since $\tilde{A}_{\mu}(x)$ and its partial derivatives are

automatically bounded on \mathbf{R}^4 .) Since H only couples nearest neighbour sites, $P_k(H^2)$ can only couple the site x to itself via a site outside of $\prod_{\mu} [x_{\mu} - de_{\mu}, x_{\mu} + de_{\mu}] \subset \mathbf{R}^4$ if $k \geq 2(d/2a)$. Therefore $[P_k(H^2)](x,x) = [P_k(\tilde{H}^2)](x,x)$ when k < d/a, and it follows that

$$\left\| \frac{1}{\sqrt{H^{2}}}(x,x) - \frac{1}{\sqrt{\tilde{H}^{2}}}(x,x) \right\| \leq \kappa \sum_{k \geq d/a}^{\infty} t^{k} \| [P_{k}(H^{2})](x,x) - [P_{k}(\tilde{H}^{2})](x,x) \|$$

$$\leq \kappa t^{d/a} \sum_{k=0}^{\infty} t^{k} \frac{1}{a^{4}}$$

$$= \left(\frac{\kappa}{1 - e^{-\theta}} \right) \frac{1}{a^{4}} e^{-\theta d/a}$$
(3.46)

This together with the fact that H is ultra-local gives (3.45).

The preceding observations allow us to conclude that $q^{U}(x) = q^{A}(x) + O(a)(x)$ in some cases where $A_{\mu}(x)$ diverges at infinity, or has singularities. An example of the former is a topologically non-trivial gauge field on the 4-torus: these can be viewed as gauge fields on \mathbb{R}^4 satisfying a periodicity condition

$$A_{\mu}(x + Le_{\nu}) = \Omega(x, \nu)A_{\mu}(x)\Omega(x, \nu)^{-1} + \Omega(x, \nu)\partial_{\mu}\Omega(x, \nu)^{-1}$$
(3.47)

where $\Omega(x,\nu)$, $\nu=1,2,3,4$, are the SU(n)-valued monodromy fields which specify the principal SU(n) bundle over T^4 . Fields $A_{\mu}(x)$ satisfying (3.47) diverge at infinity in the topologically non-trivial case. However, the requirement $||1-U(p)|| < \epsilon$ for small a still holds in this case since (i) the lattice transcript of $A_{\mu}(x)$ satisfies

$$U_{\mu}(x + Le_{\nu}) = \Omega(x, \nu)U_{\mu}(x)\Omega(x + ae_{\nu}, \nu)^{-1}$$
(3.48)

and (ii) $||U_{\mu}(y)|| = 1$ for all y, μ since $U_{\mu}(y)$ is unitary. These imply that if $x'_{\rho} = x_{\rho} + Ln_{\rho}e_{\rho}$ then $||1 - U(p_{x,\mu\nu})|| = ||1 - U(p_{x',\mu\nu})||$ for for all $n \in \mathbb{Z}^4$. It follows that $||1 - U(p)|| \sim O(a^2)$ since this is true for plaquettes in the fundamental domain $[0, L]^4 \subset \mathbb{R}^4$ because $A_{\mu}(x)$ and its partial derivatives are automatically bounded in $[0, L]^4$.

An example where the requirement $||1 - U(p)|| < \epsilon$ for small a is satisfied when $A_{\mu}(x)$ has a singularity is the following: Consider a gauge field in a singular gauge,

and such that $A_{\mu}(x)$ is pure gauge in a neighbourhood of the singularity and vanishes outside a bounded region of \mathbb{R}^4 . (Examples of topologically non-trivial gauge fields of this type are readily constructed, cf. §5.) We choose the lattices on \mathbb{R}^4 such that the singularity of $A_{\mu}(x)$ never lies on a link of any lattice (cf. §5). Then the lattice transcript $U_{\mu}(x)$ is well-defined for all lattices. For lattice plaquettes contained in the neighbourhood of the singularity we have U(p) = 1 since $A_{\mu}(x)$ is pure gauge. On the other hand, $A_{\mu}(x)$ and its partial derivatives are bounded outside the neighbourhood of the singularity (since the field vanishes outside a bounded region). Hence $||1 - U(p)|| \sim O(a^2)$.

4 Aspects of index D^U and its continuum limit in the infinite volume setting

In contrast to $q^U(x)$, the index of D^U is a problematic quantity in the infinite volume setting. It is not well-defined a priori by

$$\operatorname{index} D^{U} = \operatorname{Tr}\left(\gamma_{5}\Big|_{\ker D^{U}}\right) \tag{4.1}$$

since the null-space $\ker D^U$ may be infinite-dimensional. The same is true for the index of the continuum Dirac operator \mathscr{P}^A for general gauge field $A_\mu(x)$ on \mathbf{R}^4 . In the latter case the index exists and is equal (by the index theorem) to $Q(A) = \int_{\mathbf{R}^4} d^4x \, q^A(x)$ provided $A_\mu(x)$ converges sufficiently quickly to a pure gauge configuration at infinity. A natural conjecture in the lattice setting is therefore that index D^U , given by (4.1), exists and is equal to $a^4 \sum_x q^U(x)$ when the lattice gauge field $U_\mu(x)$ converges sufficiently quickly to a pure gauge configuration at infinity. To prove this is a challenging problem though, and in this section we will only make some first steps towards it. We investigate what happens if $q^U(x)$ is replaced by $a^4 \sum_x q^U(x)$ in the continuum limit calculation of the preceding section; this will indicate how a situation where $U_\mu(x)$ is pure gauge at infinity can result in the infinite sum $\sum_x q^U(x)$ being convergent and $a^4 \sum_x q^U(x) \to Q(A)$ in the classical continuum limit. However we do not show that

index D^U exists and is equal to $a^4 \sum_x q^U(x)$ in this case; this remains as a problem for future work.

Using the expansion (3.19) we get

$$a^{4} \sum_{x} q^{U}(x) = -\frac{1}{2} a^{4} \sum_{x} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=0}^{\infty} \text{tr}[H(G_{\sigma}V)^{k} G_{\sigma}](x, x)$$
 (4.2)

As before, the k=0 and k=1 terms give vanishing contribution. Obviously, to get $a^4 \sum_x q^U(x) \to \int_{\mathbf{R}^4} d^4x \, q^A(x) = Q(A)$ we will need the part with $k \geq 3$ to vanish in the $a \to 0$ limit as in §3. This cannot be expected to happen in general since the infinite sum \sum_{x} might not even be convergent for $k \geq 3$ part of (4.2). However, as we will now show, the sum over lattice sites does converge when $U_{\mu}(x)$ is pure gauge outside a bounded region \mathcal{R} of \mathbf{R}^4 . Furthermore, we will see that when $U_{\mu}(x)$ is the lattice transcript of a smooth continuum field $A_{\mu}(x)$ which is pure gauge outside of \mathcal{R} the $k \geq 3$ part in (4.2) does indeed vanish in the $a \rightarrow 0$ limit and we get $a^4 \sum_x q^U(x) \to Q(A)$. We can assume that \mathcal{R} is a 4-dimensional box $[-L, L]^4$. Let \mathcal{R}_d denote the larger box $[-L-d, L+d]^4$ (d>0). The strategy is to split up the sum over lattice sites in (4.2) into a finite sum $\sum_{x \in \mathcal{R}_d} (\cdots)$ and an infinite sum $\sum_{x \in \mathbf{R}^4 - \mathcal{R}_d} (\cdots)$ where the sums are over the lattice sites in \mathcal{R}_d and $\mathbf{R}^4 - \mathcal{R}_d$ respectively. The first sum is easy to deal with after noting that $a^4 \sum_{x \in \mathcal{R}_d} (\cdots) \to \int_{\mathcal{R}_d} d^4x (\cdots)$ for $a \to 0$ and using the fact that the volume $V(\mathcal{R}_d)$ is finite. We will deal with the second sum by exploiting the facts that $G_{\sigma}(z,x)$ is (exponentially) local and V(y,z) is ultra-local and vanishes in regions where U is pure gauge.

By the same calculations as in (3.20) we see that the $k \geq p$ part of the summand in (4.2) has a bound

$$\left| \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=p}^{\infty} \operatorname{tr}[H(G_{\sigma}V)^{k}G_{\sigma}](x,x) \right| \leq a^{2(p-2)}K_{p}$$
(4.3)

 $(p \geq 2)$ for some constant K_p independent of a and x. Choose an $\epsilon > 0$, then for sufficiently small a we have $a^4 \sum_{x \in \mathcal{R}_d} 1 \leq V(\mathcal{R}_d) + \epsilon \equiv V(\mathcal{R}_d)_{\epsilon}$ where $V(\mathcal{R}_d) = (2(L+d))^4$ is the volume of R_d . It follows that, for small a,

$$\left| a^4 \sum_{x \in \mathcal{R}_d} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=p}^{\infty} \operatorname{tr}[H(G_{\sigma}V)^k G_{\sigma}](x,x) \right| \leq a^{2(p-2)} K_p V(\mathcal{R}_d)_{\epsilon}$$
 (4.4)

This vanishes in the $a \to 0$ limit for p = 3, and does not diverge for p = 2. Hence when the sum over x in (4.2) is restricted to the lattice sites in \mathcal{R}_d the $k \geq 3$ part vanishes in the $a \to 0$ limit as required, and the k = 2 part remains finite. We now consider the summand in (4.2) for x outside of \mathcal{R}_d . We use

$$[H(G_{\sigma}V)^{k}G_{\sigma}](x,x) = a^{8} \sum_{y,z} [H(G_{\sigma}V)^{k-1}G_{\sigma}](x,y)V(y,z)G_{\sigma}(z,x)$$
(4.5)

By (3.18) V(y,z) = 0 for $y, z \notin \mathcal{R}$ so the sum over y and z in (4.5) can be restricted to the lattice sites in \mathcal{R} . We now apply a version of the locality argument of [37] to G_{σ} : For small a we have a-independent bounds 0 < b/2 < L < c and get a power series expansion of G_{σ} as in 3.23,

$$G_{\sigma} = \frac{1}{c + \sigma^2} \sum_{k=0}^{\infty} \left(\frac{c - L}{c + \sigma^2}\right)^k \tag{4.6}$$

which converges since $||\frac{c-L}{c+\sigma}|| < \frac{c-b/2}{c+\sigma^2} < \frac{c-b/2}{c} \equiv t < 1$. Since L only couples nearest neighbour and next to nearest neighbour sites, $L^k(x,y) = 0$ when $\sum_{\mu} |x_{\mu} - y_{\mu}|/a > 2k$. It follows that

$$||G_{\sigma}(x,y)|| \leq \frac{1}{c+\sigma^{2}} \sum_{k \geq \frac{1}{2} \sum_{\mu} |x_{\mu}-y_{\mu}|/a} \left\| \left(\frac{c-L}{c+\sigma^{2}} \right)^{k} \right\| \leq ||G_{\sigma}|| t^{\left(\frac{1}{2} \sum_{\mu} |x_{\mu}-y_{\mu}|/a \right)} \sum_{k=0}^{\infty} t^{k}/a^{4}$$

$$\leq ||G_{\sigma}|| \frac{\tilde{\kappa}}{a^{4}} \exp \left(-\frac{\theta}{2} \sum_{\mu} |x_{\mu}-y_{\mu}|/a \right)$$
(4.7)

where $t = e^{-\theta} = \frac{c - b/2}{c}$ and $\tilde{\kappa} = 1/(1 - t)$. Applying this to $G_{\sigma}(z, x)$ in (4.5) gives

$$||G_{\sigma}(z,x)|| \leq ||G_{\sigma}|| \frac{\widetilde{\kappa}}{a^4} \exp\left(-\frac{\theta}{2} \sum_{\mu} (|x_{\mu}| - L)/a\right)$$

$$(4.8)$$

since $|x_{\mu} - z_{\mu}| > |x_{\mu}| - L$ when $z \in \mathcal{R}$ and x is outside of \mathcal{R}_d . This leads to a bound on the part of (4.2) where the sum is restricted to the lattice sites in $\mathbf{R}^4 - \mathcal{R}_d$:

$$\left| a^4 \sum_{x \in \mathbf{R}^4 - \mathcal{R}_d} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=0}^{\infty} \operatorname{tr}[H(G_{\sigma}V)^k G_{\sigma}](x, x) \right|$$

$$= \left| a^4 \sum_{x \in \mathbf{R}^4 - \mathcal{R}_d} a^8 \sum_{y, z \in \mathcal{R}} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=2}^{\infty} \operatorname{tr}[H(G_{\sigma}V)^{k-1} G_{\sigma}](x, y) V(y, z) G_{\sigma}(z, x) \right|$$

$$\leq a^4 \sum_{x \in \mathbf{R}^4 - \mathcal{R}_d} \sum_{y, z \in \mathcal{R}} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=2}^{\infty} 4n ||H(G_{\sigma}V)^{k-1} G_{\sigma}|| ||V|| ||G_{\sigma}(z, x)||$$

$$\leq \left[\frac{1}{a^4} \int_{-\infty}^{\infty} \frac{d\sigma}{\pi} \sum_{k=2}^{\infty} 4n ||H(G_{\sigma}V)^k G_{\sigma}|| \right] \left(a^8 \sum_{y,z \in \mathcal{R}} 1 \right) \times
\times a^4 \sum_{x \in \mathbf{R}^4 - \mathcal{R}_d} \frac{\widetilde{\kappa}}{a^4} \exp\left(-\frac{\theta}{2} \sum_{\mu} (|x_{\mu}| - L)/a \right) \tag{4.9}$$

Since $||V|| \sim O(a^2)$ the integral is $\sim O(a^4)$. For small a we have $a^8 \sum_{y,z \in \mathcal{R}} 1 \le (V(\mathcal{R})_{\epsilon})^2$. Finally, the sum is bounded by

$$\frac{\widetilde{\kappa}}{a^8} \prod_{\mu} \left[2 \int_{L+d}^{\infty} \exp\left(-\frac{\theta}{2a} (t_{\mu} - L)\right) dt_{\mu} \right] = \widetilde{\kappa} \left(\frac{2}{a\theta}\right)^4 e^{-2\theta d/a} \tag{4.10}$$

Thus (4.9) vanishes as $O(\frac{1}{a^4}e^{-\rho/a})$ for $a \to 0$. This completes the demonstration that the sum over lattice sites in (4.2) is convergent when U is pure gauge outside a bounded region, and that the $k \geq 3$ part vanishes in the $a \to 0$ limit when U is the lattice transcript of a continuum field A which is pure gauge outside a bounded region. Furthermore, the preceding shows that in this case the non-vanishing contribution to $a^4 \sum_x q^U(x)$ in the $a \to 0$ limit comes from $a^4 \sum_{x \in \mathcal{R}_d} q_2^U(x)$ where the subscript "2" refers to the k = 2 part of (4.2). It is a straightforward technical exercise to show that this reduces to $\int_{\mathcal{R}_d} d^4x \, q^A(x)$ in the $a \to 0$ limit along the same lines as the argument for $q^U(x) \to q^A(x)$ in §3 (we omit the details). By Stokes theorem this reduces to the integral of the Chern-Simons term over the boundary $\partial \mathcal{R}_d$, which calculates the winding number of the map from $\partial \mathcal{R}_d \cong S^3$ to SU(n) corresponding to the pure gauge configuration, and this number is precisely Q(A).

Clearly the preceding arguments can be generalised from the case where U and A are pure gauge outside a bounded region of \mathbb{R}^4 to cases where the fields converge sufficiently quickly to pure gauge configurations at infinity. To determine a precise criterion for what is "sufficiently quickly" is a non-trivial problem though, and is beyond the scope of this paper. In the continuum the criterion is that $F_{\mu\nu}(x)$ should vanish quickly enough at infinity to be square-integrable [12]. We speculate that a similar condition will suffice in the lattice setting.

5 Classical continuum limit of index D^U starting in a finite volume lattice setting and taking an infinite volume limit

In this section we consider index D^U in the lattice setting (IV) of §3. The volume is finite, hence the space of lattice spinor fields is finite-dimensional and index D^U is well-defined from the beginning. The boundary condition on the lattice gauge field is that it is trivial at the boundary (i.e. the Dirichlet condition). Then the covariant finite difference operators ∇^{\pm}_{μ} map both the spaces of periodic and antiperiodic lattice spinor fields to themselves, and in both cases satisfy $(\nabla_{\mu}^{\pm})^* = -\nabla_{\mu}^{\mp}$, so H is hermitian and D^U is well-defined. We can therefore take either periodic or antiperiodic boundary conditions on the lattice spinor fields. We consider the case where U is the lattice transcript of a continuum gauge field $A_{\mu}(x)$ in a singular gauge and vanishing outside a bounded region of \mathbb{R}^4 . The singularity allows A to be topologically non-trivial (i.e. $Q(A) \neq 0$) while the latter requirement ensures that $U_{\mu}(x)$ is trivial outside of a bounded region of \mathbb{R}^4 and therefore satisfies the Dirichlet boundary condition when the volume is sufficiently large. We will show that index \mathcal{D}^U reduces to the continuum topological charge Q(A) in the classical continuum limit defined by first taking the infinite volume limit and then taking $a \to 0$. Since index D^U and Q(A)are both integers, it then follows that index $D^U = Q(A)$ for all sufficiently large lattices with sufficiently small lattice spacings. This demonstrates analytically that index D^U is able to capture the topological data of the continuum field A (complementing previous numerical results which we discuss later). The present setting is arguably the simplest in which such a result can be analytically demonstrated. The same result holds in the finite volume torus setting [15]; however, the argument is less simple than what follows. 12

Let $A_{\mu}(x)$ be an SU(n) gauge field on \mathbf{R}^4 which may be singular at the origin but is smooth everywhere else and vanishes outside a bounded region. Furthermore we require that A is pure gauge in a neighbourhood of the origin. Examples of such fields

¹²In fact the result derived in this section can also be derived quite straightforwardly as an application of the result for the finite volume torus case [40].

are readily obtained as follows. Take a smooth map $\phi: S^3 \to SU(n)$ with degree Q and define $\tilde{\phi}: \mathbf{R}^4 - \{0\} \to SU(n)$ by $\tilde{\phi}(y,t) = \phi(y)$, $y \in S^3$, $t \in \mathbf{R}_+$, where we are identifying $\mathbf{R}^4 - \{0\}$ with $S^3 \times \mathbf{R}_+$ in the obvious way. Choose a smooth real function $\lambda(x)$ on \mathbf{R}^4 equal to 1 in a neighbourhood of the origin and vanishing outside a bounded region. Then the field

$$A_{\mu}(x) = \lambda(x)\widetilde{\phi}(x)\partial_{\mu}\widetilde{\phi}(x)^{-1} \tag{5.1}$$

has topological charge Q, is singular at the origin, pure gauge in a neighbourhood of the origin and vanishes outside the bounded region.

We take the finite hyper-cubic lattice in \mathbb{R}^4 to have spacing a and sites

$$\{x = a(n_1 - 1/2, n_2 - 1/2, n_3 - 1/2, n_4 - 1/2)\} \mid n \in \mathbf{Z}^4 - N < n_{\mu} \le N \}$$
 (5.2)

Note that the origin $0 \in \mathbf{R}^4$ never lies on a lattice link, so the lattice transcript $U_{\mu}(x)$ of $A_{\mu}(x)$ (given by (3.9)) is well-defined for all a and N.

Our aim in this section is to show that, with either periodic or anti-periodic boundary conditions on the lattice spinor fields,

$$\lim_{a \to 0} \lim_{N \to \infty} \operatorname{index} D^U = I(m_0)Q(A)$$
(5.3)

where $I(m_0)$ is given by the table (3.43) (in particular $I(m_0) = 1$ for the physical values $0 < m_0 < 2$). The techniques used to derive this have already been developed in the earlier parts of this paper, so we will be economical with the details in the following.

Proof of (5.3). We start from the formula

index
$$D^{U} = a^{4} \sum_{x} q^{U}(x)$$
 , $q^{U}(x) = -\frac{1}{2} \text{tr} \frac{H}{\sqrt{H^{2}}}(x, x)$ (5.4)

where the sum is over the lattice sites (5.2). Choose L sufficiently large so that $A_{\mu}(x)$ vanishes for all x outside of $\mathcal{R} = [-L, L]^4$. Choose d > 0 and set $\mathcal{R}_d = [-L-d, L+d]^4$ as before. By the argument at the end of §3 we have $||1-U(p)|| \sim O(a^2)$. This leads to a lower bound $H^2 > b > 0$ for small a, with b independent of both a and N. Therefore

the previous bounds such as $||V|| \leq a^2 K$ and L > b/2 continue to hold in the present setting and can be chosen independent of N. We henceforth restrict to the (sufficiently small) lattice spacings a for which these bounds are satisfied. Furthermore, for given a we restrict to the (sufficiently large) N's for which aN > L + d so that the lattice (specified by (5.2)) covers \mathcal{R}_d . The $N \to \infty$ limit of the part of (5.4) where the sum is restricted to sites outside of \mathcal{R}_d has the bound

$$\lim_{N \to \infty} \left| a^4 \sum_{x \notin \mathcal{R}_d} q^U(x) \right| \le a^4 \sum_{x \in \mathbf{R}^4 - \mathcal{R}_d} |q^U(x)| \tag{5.5}$$

where the sum on the right-hand side is over all lattice sites outside of $\mathbf{R}^4 - \mathcal{R}_d$ in the extension of the lattice to the whole of \mathbf{R}^4 . This vanishes in the $a \to 0$ limit by a similar argument to the one based on (4.9)–(4.10) in the previous section. Thus in deriving (5.3) it suffices to restrict the sum in (5.4) to the lattice sites in \mathcal{R}_d .

Now choose L'>0 small enough that $[-L',L']^4$ is contained within the neighbourhood of the origin in \mathbf{R}^4 where $A_{\mu}(x)$ is pure gauge. Choose e>0 so that e< L' (e.g. e=L'/2) and set $\mathcal{R}_e=[-L'+e,L'-e]^4$. By a straightforward adaptation of the argument based on (4.9)–(4.10) we find that the part of (5.4) where the sum is restricted to the lattice sites in \mathcal{R}_e vanishes in the $a\to 0$ limit after taking $N\to\infty$. Thus in deriving (5.3) it suffices to restrict the sum in (5.4) to the lattice sites which are contained in $\mathcal{R}_d-\mathcal{R}_e$, i.e. it suffices to show

$$\lim_{a \to 0} \lim_{N \to \infty} a^4 \sum_{x \in \mathcal{R}_d - \mathcal{R}_e} q^U(x) = I(m_0) Q(A).$$

$$(5.6)$$

Using the locality of $q^U(x)$ in the gauge field (cf. §3), arguments similar to those in §3 lead to an analogue of (3.36): In the present setting the lattice delta-function has the Fourier expansion (analogue of (3.31))

$$\delta_x = \sum_{-\pi \le k_u \le \pi} \frac{\Delta^4 k}{(2\pi)^4} e^{-ikx/a} \phi_k \tag{5.7}$$

where $\phi_k(y) := e^{iky/a}$ is the lattice plane wave field with momentum k; the domain of the lattice 4-momentum (i.e. the summation domain in (5.7)) in the case of periodic or anti-periodic boundary conditions is

periodic:
$$k_{\mu} \in \frac{\pi}{N} \{-N, -N+1, \dots, N-1\}$$
 (5.8)

anti-periodic:
$$k_{\mu} \in \frac{\pi}{N} \{-N + 1/2, -N + 3/2, \dots, N - 1/2\}$$
 (5.9)

and $\Delta^4 k = \pi^4/(2N)^4$ = the "volume per k" in lattice momentum space. Using this we find, analogously to (3.36),

$$q^{U}(x) = I(r, m_0, N)q^{A}(x) + O(a)(x)$$
(5.10)

where

$$I(r, m, N) = \frac{-3r}{8\pi^2} \sum_{-\pi < k_{\mu} < \pi} \Delta^4 k \,\lambda(k; r, m) \,. \tag{5.11}$$

with $\lambda(k; r, m)$ given by (3.35). The N-dependence is in the summation domain: (5.8) or (5.9). In the $N \to \infty$ limit $\Delta^4 k \to d^4 k$ and we see from (5.8)–(5.9) that in both the periodic and anti-periodic cases $\lim_{N\to\infty} I(r, m, N) = I(r, m) = I(m)$ (cf. §3 for the last equality). It follows that

$$\lim_{a \to 0} \lim_{N \to \infty} a^4 \sum_{x \in \mathcal{R}_d - \mathcal{R}_e} q^U(x) = I(m_0) \int_{\mathcal{R}_d - \mathcal{R}_e} d^4 x \, q^A(x) \,. \tag{5.12}$$

By Stokes theorem this reduces to the difference of the integrals of the Chern-Simons term over the boundaries $\partial \mathcal{R}_d$ and $\partial \mathcal{R}_e$. The former vanishes since $\partial \mathcal{R}_d$ lies in the region where $A_{\mu}(x)$ vanishes. The latter gives the winding number of the map from $\partial \mathcal{R}_e \cong S^3$ to SU(n) corresponding to the pure gauge configuration, and this is precisely Q(A) in the present singular gauge case. This completes the proof of (5.3). Comparison with numerical results. Our result (5.3) implies that for $0 < m_0 < 2$ the index of D^U reproduces the continuum topological charge Q(A) when N is sufficiently large and a is sufficiently small, and that the index reduces to $I(m_0)Q(A)$ for general $m_0 \notin \{0, 2, 4, 6, 8\}$. This is something which can be investigated numerically, and in fact numerical investigations in closely related lattice setups have already been carried out. In finite volume settings such as the present one (where the space of lattice spinor fields is finite-dimensional) the index of D^U equals the spectral flow of $-H_m$ as m increases from any negative value to m_0 [4]. This follows from (2.11): index $D^U = -\frac{1}{2} \text{Tr}(\frac{H}{|H|})$ (recall $H = H_{m_0}$) and the fact that the spectrum of H_m is symmetric and without zero for m < 0 [1, 2, 14]. In [14] the spectral flow of H_m was studied numerically for various smooth SU(2) instanton fields on a finite lattice.¹³

¹³The spectral flow of H_m for various lattice gauge fields was first studied numerically in [34].

One of the situations considered in [14] was instanton field in a singular gauge with anti-periodic boundary conditions on H_m . I.e. anti-periodic boundary conditions on the lattice spinor fields, and the lattice transcript of the instanton field modified at the boundary links to make it anti-periodic, so that H_m is a well-defined hermitian operator on these spinor fields. This is a minor modification since the localisation radius of the singular gauge instanton in [14] was well within the region covered by the lattice, i.e. it was almost vanishing at the boundary of the lattice. This is similar to the setup that we have considered in this section: the singular gauge instanton can be approximated by a gauge field $A_{\mu}(x)$ with singularity which is pure gauge in a neighbourhood of the singularity and vanishes outside a bounded region. The results of [14] are compatible with ours: They numerically determined the crossings of the origin by low-lying eigenvalues $\lambda(m)$ of H_m for 0 < m < 2 and found precisely one crossing, which occurred reasonably close to zero (at $m \approx 0.5$), and the slope of $\lambda(m)$ at the crossing was such that the spectral flow of $-H_m$ from this crossing agreed with the topological charge of the instanton (cf. Fig. 7 of [14]). This can be equivalently expressed as index $D^U = Q(A)$ for $0.6 < m_0 < 2$. This is an approximative numerical confirmation of our analytic result in this section. Numerical studies of the spectral flow of H_m in other situations in [34, 14], or equivalently, numerical studies of index D^U as a function of m_0 [32], are also compatible with the classical continuum limit result for index D^U that we have shown for the particular situation considered in this section (and which has also been analytically shown in the finite volume torus case in [15]): For "sufficiently smooth" lattice gauge fields it was found that (i) crossings of the origin by eigenvalues $\lambda(m)$ of $-H_m$ occur close to m=0,2,4,6,8; if the net spectral flow from crossings close to m=0 is Q then the net spectral flows from crossings close to m=2,4,6,8 are, respectively, $-4Q\,,\,6Q\,,\,-4Q\,,\,Q\,,$ and (ii) if Uis the lattice transcript of a continuum field A then Q = Q(A). (A situation where this does not hold is when U is the lattice transcript of an instanton field in a regular gauge and anti-periodic boundary conditions are imposed on H_m [14]. This is to be expected though, since in this case the instanton field is approximately pure gauge at the boundary of the region covered by the lattice, and is therefore not close to

satisfying anti-periodic boundary conditions.)

6 Conclusion

In this paper we have rigorously verified that the index density/axial anomaly for the overlap Dirac operator has the correct classical continuum limit in the infinite volume lattice setting, and in a finite volume setting where the the continuum limit involves an infinite volume limit. The only condition on the continuum field $A_{\mu}(x)$ required to establish this result is that the plaquette variable of its lattice transcript satisfies

$$||1 - U(p)|| < \epsilon(m_0) \tag{6.1}$$

for sufficiently small lattice spacing a, with $\epsilon(m_0)$ small enough to guarantee the existence of a lower bound $b(m_0) > 0$ on H^2 [37, 38, 39], which in turn guarantees the locality of D^U in the gauge field [37]. The locality plays a central role in our arguments. The condition (6.1) is automatically satisfied for small a when $A_{\mu}(x)$ and its first and second order partial derivatives are bounded on \mathbb{R}^4 , since it can then be shown that $||1-U(p)|| \sim O(a^2)$. However, using the locality property we have seen that the condition is satisfied for more general gauge fields, including some cases where the field has a singularity or diverges at infinity. In particular our results cover the case of topologically non-trivial gauge fields on \mathbb{R}^4 . This case was was not covered by previous works on this topic (cf. the discussion in §2). Our approach was to decompose the operator H^2 appearing in the overlap Dirac operator $D = \frac{1}{a} \left(1 + \gamma_5 \frac{H}{\sqrt{H^2}} \right)$ into $H^2 = L - V$ where all the gamma-matrices are contained in V and $||V|| \sim O(a^2)$. Then, starting with an integral representation for the inverse square root, we expand $1/\sqrt{H^2}$ in powers of V. It was found that only the term of order 2 in V contributes to the index density $q^U(x) = -\frac{1}{2} \operatorname{tr} \frac{H}{\sqrt{H^2}}(x,x)$ in the classical continuum limit, and the contribution from this term was shown to reduce to the continuum density $q^A(x)$ (for $0 < m_0 < 2$).

The expansion of the overlap Dirac operator in powers of V is of interest in its own right and may have other useful applications. Unlike an expansion in powers

of the gauge field, such as the one given in [25], which requires the gauge field to be small (i.e. link variables close to 1), our expansion only requires a bound of the form (6.1) to hold, i.e. plaquette variables close to 1, and can therefore be used for topologically non-trivial gauge configurations.

Subtleties/pitfalls which arise when considering the index of the overlap Dirac operator in the infinite volume setting were discussed (§4). In this case the index is generally ill-defined a priori. Formally we have index $D^U = a^4 \sum_x q^U(x)$, and we pointed out that the previous result $q^U(x) \to q^A(x)$ for the index density does not imply $a^4 \sum_x q^U(x) \to \int_{\mathbf{R}^4} d^4 q^A(x) = Q(A)$ in general (as was erroneously claimed in [13]): The argument was seen to break down due to the infinite sum over lattice sites in $\sum_x q^U(x)$. We showed however that $a^4 \sum_x q^U(x) \to \int_{\mathbf{R}^4} d^4 x \, q^A(x) = Q(A)$ in the case where $A_\mu(x)$ is pure gauge outside a bounded region of \mathbf{R}^4 . This is a first step towards showing the following conjecture: index D^U is well-defined and equal to $a^4 \sum_x q^U(x)$ when the lattice field $U_\mu(x)$ converges sufficiently quickly to a pure gauge configuration at infinity, and in this case index $D^U \to Q(A)$ in the classical continuum limit. This conjecture (the continuum analogue of which is known to be true [12]) remains as an interesting problem for future work.

In order to verify that the index of the overlap Dirac operator really is able to capture topological information we considered (in §5) a finite volume lattice setting where index D^U is well-defined from the beginning, namely when U is the lattice transcript of a continuum field $A_{\mu}(x)$ which is in a singular gauge and vanishes outside a bounded region. (For technical reasons we also required $A_{\mu}(x)$ to be pure gauge in a neighbourhood of the singularity. We described how such gauge fields can be readily constructed from maps $\phi: S^3 \to SU(n)$.) We showed that $\lim_{a\to 0} \lim_{N\to\infty} \operatorname{index} D^U = Q(A)$, where $N\to\infty$ is the infinite volume limit. This result complements a previous numerical result obtained in a similar setting in [14].

For the finite volume 4-torus case the results $\lim_{a\to 0} q^U(x) = q^A(x)$ and $\lim_{a\to 0} \operatorname{index} D^U$ = Q(A) are established for general gauge fields in a separate paper [15]. In that case the result for $q^U(x)$ immediately implies the result for index D^U since the volume is finite. However, the finite volume causes additional technical complications in the calculation of $\lim_{a\to 0} q^U(x)$; we refer to [15] for the details.

Finally we mention that, while the settings we have considered were all in 4 dimensions, everything generalises straightforwardly to arbitrary even dimension 2m, with gauge group SU(n) for $2m \geq 4$ and gauge group U(1) in the 2-dimensional case. In the latter case our result $q^U(x) \to q^A(x)$ confirms a numerical result in §10 of [2] where the axial anomaly $\mathcal{A}^U(x) = 2iq^U(x)$ was numerically determined and compared to the continuum axial anomaly for a particular topologically non-trivial U(1) gauge field on the 2-torus.¹⁴

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¹⁴I thank Herbert Neuberger for pointing this out (cf. [41]).

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